Indian Statistical Institute, Bangalore
B. Math. Second Year
First Semester - Optimization

Duration : 3 hours Date : November 22, 2017

Semester Exam **Total marks: 50**

Section I: Answer any four and each question carries 6 marks.

1. Prove QR-decomposition for full column rank matrices and prove the decomposition is unique if R is required to have positive entries on the diagonal.

   Solution: Let A be an $m \times n$ matrix with $m \geq n$ and rank of A is n. Let a_i be the ith column of A. Then $\{a_1, ..., a_n\}$ is a linearly independent set in $\mathbb{R}^m$.

   Let
$$u_1 = a_1, u_2 = \frac{< u_1, a_2 >}{||u_1||^2} u_1, \tag{1}$$
$$u_i = a_i - \sum_{k=1}^{i-1} \frac{< u_k, a_i >}{||u_k||^2} u_k. \tag{2}$$
   Let $\bar{e_i} = \frac{u_i}{||u_i||}$ for all $i \leq i \leq n$. Then $\{\bar{e_1}, ..., \bar{e_n}\}$ is an orthonormal set, such that $\text{span}\{\bar{e_1}, ..., \bar{e_n}\} = \text{span}\{a_1, ..., a_n\}$. Let $Q = [\bar{e_1}.....\bar{e_n}]$. Then Q is an $m \times n$ orthogonal matrix.

   Let $R = \begin{pmatrix} < \bar{e_1}, a_1 > & < \bar{e_1}, a_2 > & ... & \bar{e_1}, a_n > \\ 0 & < \bar{e_2}, a_2 > & ... & < \bar{e_2}, a_n > \\ ... & ... & ... & ... \\ 0 & 0 & ... & < \bar{e_n}, a_n > \end{pmatrix}$

   Then $A = QR$.

   Now we show that the QR-decomposition is unique if R is required to have positive entries on the diagonal. Let
$$A = Q_1 R_1 = Q_2 R_2.$$
   Now
$$R_1^t R_1 = R_1^t Q_1^t Q_1 R_1 = (Q_1 R_1)^t Q_1 R_1 = A^t A.$$
   Similarly, $R_2^t R_2 = A^t A$. Now
$$R_2 R_1^{-1} = (R_2^t)^{-1} R_1^t = ((R_2 R_1^{-1})^t)^{-1}.$$

$R_2R_1^{-1}$ is upper triangular and $((R_2R_1^{-1})^t)^{-1}$ is lower triangular. Hence $R_2R_1^{-1} = D$, where D is a diagonal matrix. But again $D = (D^t)^{-1} = D^{-1}$, which implies that D is the identity matrix. Hence $R_1 = R_2$ and consequently $Q_1 = Q_2$.

2. If $A = UDV^t$ is the singular value decomposition, prove that $Ax = b$ has a solution if and only if $b \perp U_i$ for all $i > k$ where k is the rank of A.

   Solution: Let σ_1, σ_2, ... , σ_k be the singular values of A. Then the non-zero entries of the diagonal matrix D are σ_1, σ_2, ... , σ_k respectively.

   Now
$$Ax = b \Leftrightarrow UDV^tx = b \Leftrightarrow DV^tx = U^tb.$$

   As D is a diagonal matrix with first k entries being non-zero and
$$U^tb = \begin{pmatrix} < U_1, b > \\ < U_2, b > \\ . \\ . \\ < U_m, b > \end{pmatrix},$$
   it follows that $Ax = b$ has a solution if and only if $b \perp U_i$ for all $i > k$.

3. Prove that Spr(A) has algebraic multiplicity one for a nonnegative irreducible matrix A.

   Solution: Let r =Spr(A) and $\phi(\lambda) = det(\lambda I - A)$, where I is the identity matrix. it is enough to show that $\phi'(r) \neq 0$. Let $(\lambda I - A)_{kl}$ be the matrix obtained from $(\lambda I - A)$ by deleting kth row and lth column. Let
$$c_{ij}(\lambda) = (-1)^{i+j} det[(\lambda I - A)_{ij}] \,\forall\, i, j.$$

   Let $c(\lambda) = (c_{ij}(\lambda))$. Then
$$[(\lambda I - A)c(\lambda] = \phi(\lambda)I = c(\lambda)(\lambda I - A).$$

   Differentiating with respect to λ, we get
$$c'(\lambda)(\lambda I - A) + c(\lambda) = \phi'(\lambda)I.$$

   Putting $\lambda = r$, we get
$$c'(r)(rI - A) + c(r) = \phi'(r)I.$$

Let $T = \begin{pmatrix} A_1 & O_{n-1\times 1} \\ O_{1\times n-1} & 0 \end{pmatrix}$, where A_1 is obtained from A by deleting the nth row and nth column.Then $T \geq 0$ and $A \geq T$. Moreover T is not irreducible. Hence $T \neq A$. Since $A \geq T$, $r =$Spr(A) $\geq$Spr(T) and $(rI - T)$ is one-one. Therefore, $c(r) \neq 0$. Let $u \in C_A \cap B$ such that $A(u) = ru$. Now

$$c'(r)(rI - A)(u) + c(r)u = \phi'(r)I(u)$$

$$\Rightarrow c(r)(u) = \phi'(r)(u)$$

$$\Rightarrow \sum c(r)_{nj}u_j = \phi'(r)u_n.$$

We have $u \in C_A \cap B$ and $A(u) = ru$, which implies that

$$(rI - A)c(r) = \phi(r)I = O_{n\times n}.$$

Hence $(rI - A)c_j = 0$, where c_j is the jth column of $c(r)$. So, $A(c_j) = rc_j$ for all j. Therefore, all column vectors of $c(r)$ are eigenvectors of A corresponding to the eigenvalue r. Since the eigenvalue r has geometric multiplicity 1, it follows that each column of $c(r)$ is a multiple of u.

Now A^t is also non-negative and irreducible and Spr(A) = Spr$(A^t) = r$. There exists $v \in B$ such that $A^t(v) = rv$ and $v_i > 0$ for all i. It follos as above that all rows of $c(r)$ are constant multiples of v. Therefore, $c(r)^t_n$ is a non-zero vector with all components having same sign. Hence

$$\sum_{j=1}^{n} c_{nj}(r)u_j \neq 0 \;\; (u_j \neq 0).$$

Then it follows that $(c(r)u)_n \neq 0$. So, $\phi'(r)u = c(r)u \neq 0$, which implies that $\phi'(r) \neq 0$.

4. Solve by simplex method

$$\begin{array}{cc} \text{Maximize} & 9x_1 + 10x_2 \\ \text{subj} & x_1 + 2x_2 \leq 8 \\ & 5x_1 + 2x_2 \leq 16 \\ & x \geq 0. \end{array}$$

Solution: Starring with the initial simplex tableau and applying simplex method, we get

$$\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & z & \\ \hline 1 & 2 & 1 & 0 & 0 & 8 \\ 5 & 2 & 0 & 1 & 0 & 16 \\ \hline -9 & -10 & 0 & 0 & 1 & 0 \end{array} \tag{3}$$

Applying the operation $\frac{1}{2}R_1$ we get

x_1	x_2	x_3	x_4	z	
$1/2$	1	$1/2$	0	0	4
5	2	0	1	0	16
-9	-10	0	0	1	0

(4)

Then applying $R_2 - 2R_1$ and $R_3 + 10R_1$, we get

x_1	x_2	x_3	x_4	z	
$1/2$	1	$1/2$	0	0	4
4	0	-1	1	0	8
-4	0	5	0	1	40

(5)

Again $\frac{1}{4}R_2$ gives

x_1	x_2	x_3	x_4	z	
$1/2$	1	$1/2$	0	0	4
1	0	$-1/4$	$1/4$	0	2
-4	0	5	0	1	40

(6)

Then applying $R_1 - \frac{1}{2}R_2$ and $R_3 + 4R_2$, we get

x_1	x_2	x_3	x_4	z	
0	1	$5/8$	$1/8$	0	3
1	0	$-1/4$	$1/4$	0	2
0	0	4	1	1	48

(7)

Hence the optimal solution is $x_1 = 2$, $x_2 = 3$ and the maximum value is 48.

5. Solve the following game: Ruby conceals either a Rs. 1 coin or Rs. 2 coin in her hand; Charm guesses 1 or 2, winning the coin if he guesses the number.

   Solution: The matrix corresponding to this game is

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

. Therefore, it does not have any saddle point, so it is a non-strictly determind game. Hence the optimal strategies are given by, $p^* = (p_1^*, p_2^*)^t$ and $q^* = (q_1^*, q_2^*)^t$, where

$$p_1^* = \frac{d-c}{(a-b)-(c-d)} = \frac{2}{3},$$

$$p_2^* = \frac{a-b}{(a-b)-(c-d)} = \frac{1}{3},$$
$$q_1^* = \frac{d-b}{(a-b)-(c-d)} = \frac{2}{3},$$
$$q_2^* = \frac{a-c}{(a-b)-(c-d)} = \frac{1}{3}.$$

6. Prove the existence and uniqueness of minimum norm least square solution to $Ax = b$.

   Solution: Let A be an $m \times n$ matrix and $A = UDV^t$ be a singular value decomposition of A. $D = diag(\sigma_1, ..., \sigma_r)$, $r = \rho(A)$. Then the minimum norm least square solution to $Ax = b$ is given by $\bar{x} = VD^+U^tb$, where $D^+ = diag(\frac{1}{\sigma_1}, ..., \frac{1}{\sigma_r})$.

   $$||Ax - b||^2 = ||UDV^tx - b||^2 = ||DV^tx - U^tb||^2$$

   . Let $y = V^tx$, $c = U^tb$, then $||y|| = ||x||$. Now

   $$min_{x\in\mathbb{R}^n}||Ax - b||^2 = min_{y\in\mathbb{R}^n}||Dy - c||^2.$$

   $$||Dy - c||^2 = \sum_{i=1}^{r}(\sigma_i y_i - c_i)^2 + \sum_{i>r} c_i^2 \geq \sum_{i>r} c_i^2$$

   . Define the vector $\bar{y}$ as follows: the ith entry is $\frac{c_i}{\sigma_i}$ if $i \leq r$ and the ith entry is 0 if $i > r$. Then

   $$||D\bar{y} - c||^2 = \sum_{i>r} c_i^2 \leq ||Dy - c||^2$$

   for all $y \in \mathbb{R}^n$. Now

   $$||Dy - c||^2 = \sum_{i>r} c_i^2 \Rightarrow y_i = \frac{c_i}{\sigma_i}\forall i \leq r \Rightarrow ||y|| \geq ||\bar{y}||.$$

   Hence the proof of existence and uniqueness.

Section II: Answer any two and each question carries 13 marks.

1. (a) Let s_i be the i-th singular value of A. Prove that $s_i \leq ||A - B||$ for any matrix B with rank $(B) < i$ (*Marks: 7*).

   Solution: Let A be of order $p \times q$ and $A = VDU^*$ be the singular value decomposition of A, where $D = diag(s_1, ..., s_r)$. Let B be a $p \times q$ matrix such that rank$(B) < i$. Let u_i denote the ith column of U and v_i denote the ith column of V. Let

$$U_1 = [u_1, ..., u_i, 0, ..., 0]$$

and

$$V_1 = [v_1, ..., v_i, 0, ..., 0].$$

Since rank$(V_1^* B U_1) \leq$rank$(B) \leq i-1$, there exists $c \in \mathbb{C}^q$ such that $||c|| = 1$, $c \in$span$\{e_1, ..., e_i\}$ and

$$V_1^* B U_1(c) = 0$$

$$\Rightarrow \sum_{j=1}^{i} c_j < B(u_j), v_k >= 0 \forall 1 \leq k \leq i$$

$$\Rightarrow \left\langle B(\sum_{j=1}^{i} c_j u_j), v_k \right\rangle = 0, 1 \leq k \leq i.$$

Now let $x = \sum_{j=1}^{i} c_j u_j$, then $B(x) \perp v_k \forall 1 \leq k \leq i$. We know $A = \sum_{j=1}^{r} s_j v_j u_j^*$. Then

$$A(x) = \sum_{l,j} s_l c_j v_l u_l^* u_j = \sum_{j=1}^{i} s_j c_j v_j.$$

Now

$$||A(x) - B(x)||^2 = ||\sum_{j=1}^{i} s_j c_j v_j - \sum_{j=i+1}^{p} \langle B(x), v_j \rangle v_j||$$

$$= \sum_{j=1}^{i} |s_j|^2 |c_j|^2 + \sum_{j=i+1}^{p} |\langle B(x), v_j \rangle v_j|^2.$$

Hence

$$||A(x) - B(x)|| \geq s_i.$$

As $x = \sum_{j=1}^{i} c_j u_j$ and $||c|| = 1$, it follows that $||x|| = 1$. Hence $s_i \leq ||A - B||$.

(b) For $A = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & 0 \end{pmatrix}$, use Perron-Frobenius theory to compute $\lim_{n\to\infty} A^n$.

Solution: A is a primitive (since A^2 is a positive matrix) and stochastic matrix. So, 1 is the dominant eigenvalue for A and A^t as well. Also let $u = (1, 1, 1)^t$

and $v = (1/4, 3/8, 3/8)$. Then it is easy to see that $A(u) = u$ and $A^t(v) = v$. We also note that $< u, v >= 1$. Hence, by perron-Frobenius theory,

$$\lim_{n\to\infty} A^n = uv^t = \begin{pmatrix} 1/4 & 3/8 & 3/8 \\ 1/4 & 3/8 & 3/8 \\ 1/4 & 3/8 & 3/8 \end{pmatrix}.$$

P.T.O

2. (a) Let $A \geq 0$ be a primitive matrix, v and u be positive eigenvectors for A and A^t with eigenvalue λ =Spr(A). If $u^t v = 1$ and $r =$ Spr(A), prove that $(\frac{A}{r})^n \to vu^t$ exponentially (*Marks: 6*).

Solution: Let $M = vu^t$. Note that

$$u^t M = u^t vu^t = u^t$$

and

$$M(v) = vu^t v = v.$$

Now let $W =< u >^{\perp}$. For $w \in W$

$$< u, A(w) >=< A^t(u), w >= \lambda < u, w >= 0.$$

Hence $A(w) \in W$ and therefore, W is an invariant subspace for A. Also for $w \in W$,

$$M(w) = vu^t w = 0.$$

Let T be an operator such that $T(w) = A(w)$ for $w \in W$ and $T(u) = 0$. Then Spr$(T) <$ Spr(A). Let β =Spr(T) and $\varepsilon > 0$ be such that $\beta + \varepsilon < \lambda$. Then we know that there exists $N \in \mathbb{N}$ such that

$$||T^n||^{1/n} \leq \beta + \varepsilon \ \forall \ n \geq N.$$

$$i.e. \ ||T^n|| \leq (\beta + \varepsilon)^n \ \forall \ n \geq N.$$

Now for $w \in W$,

$$|| \left(\frac{A}{\lambda}\right)^n (w)|| = \left(\frac{1}{\lambda}\right)^n ||A^n(w)|| = \left(\frac{1}{\lambda}\right)^n ||T^n(w)||$$

$$\leq \frac{1}{\lambda^n}||T^n||||w|| \leq \left(\frac{\beta + \varepsilon}{\lambda}\right)^n ||w||,$$

which tends to 0 exponentially as $n \to \infty$. This implies that $\left(\frac{A}{\lambda}\right)^n \to M = vu^t$ exponentially, since $\left(\frac{A}{\lambda}\right)^n (v) = v = M(v)$.

(b) Let A and B be two matrices such that $b_{ij} = a_{ij} + r$. Then show that two strategy vectors p and q are optimal for A if and only if they are optimal for B and value of the game B is value of game A plus r.

Solution: Let A and B be $m \times n$ matrices. For any $p = (p_1, ..., p_m)^t$ and $q = (q_1, ..., q_n)$,

$$p^t Bq = \sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij} p_i q_j = \sum_{i=1}^{m} \sum_{j=1}^{n} (a_{ij} + r) p_i q_j = p^t Aq + r.$$

Now let p^* and q^* be an optimal strategy for B. Then

$$pBq^* \leq v(B) \leq p^*Bq \;\; \forall \;\; p \in \mathbb{R}^m, q \in \mathbb{R}^n.$$

Hence by the calculation above,

$$pAq^* + r \leq v(B) \leq p^*Aq + r \;\; \forall \;\; p \in \mathbb{R}^m, q \in \mathbb{R}^n,$$

equivalently,

$$pAq^* \leq v(B) - r \leq p^*Aq \;\; \forall \;\; p \in \mathbb{R}^m, q \in \mathbb{R}^n.$$

This shows that $v(A) = v(B) - r$. We have also shown that p^* and q^* is also an optimal strategy for A as well. By a similar argument as above it can also be shown that an optimal strategy for A is also an optimal strategy for B.

3. (a) Describe and justify a method to avoid anticycling in LP (*Marks: 6*).

   (b) State and prove a necessary and sufficient condition in terms of the matrix entries for a 2×2- matrix game to be non-strictly determined.

   Solution: Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ be a 2×2 matrix corresponding to a matrix game. The game is non-strictly determind if and only if none of the matrix entries is a saddle point. We show that if there is a saddle point then the game is strictly determind, from which the assertion follows. Let a_{12} be a saddle point. Then $a_{12} \leq a_{11}$ and $a_{12} \geq a_{22}$. Then $e_1^t A e_j \geq a_{12}$ for $j = 1, 2$ and $e_i^t A e_2 \leq a_{12}$ for $i = 1, 2$. This shows that $v(A) = a_{12}$ with e_1 and e_2 determining an optimal strategy.

   (c) Solve the $n \times n$-game $A = I_n$ (*Marks: 3*).

   Solution: We claim that $p = q = (\frac{1}{n}, ..., \frac{1}{n})^t$ will provide an optimal strategy for this game. If $r = (r_1, ..., r_n)^t$ is any other vector in $\mathbb{R}^n$ with $v_1 + ... + v_n = 1$, then

   $$p^t A v = < p, v > = \frac{1}{n}.$$

   Similarly,

   $$v^t A q = < v, q > = \frac{1}{n}.$$

   This shows that $p = q = (\frac{1}{n}, ..., \frac{1}{n})^t$ defines an optimal strategy for this game and $v(A) = v(I_n) = \frac{1}{n}$.